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THE DE RHAM PRODUCT DECOMPOSITION

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1. Introduction

The main purpose of this paper is to present a simple proof of the de Rham product decomposition theorem for semi-Riemannian manifolds. In fact, we extend the theorem to the case of metric connections with torsion. As a by-product of our methods, we also obtain a simple proof of the Ambrose-Hicks theorem on parallel translation of curvature.

The original theorem, for Riemannian manifolds, appeared in de Rham [5]. Another proof appeared in Kobayashi and Nomizu [3, Vol. I, pp. 179–193], which is also the general reference for this paper (see also Vol. II, p. 331). The semi-Riemannian case is due to H. Wu [7]. Our proof uses an elegant method of constructing Riemannian covering maps due to B. O'Neill [4]. The advantage of using this construction lies in the fact that homotopy considerations can be dispensed with, being absorbed in the theory of covering spaces.

In § 2 we give a brief exposition of O'Neill's construction, adapted to the affine (or semi-Riemannian) case. We then extend his simple proof of the Ambrose theorem on parallel translation of curvature to the affine case, due to Hicks [1].

In § 3 we use the same construction to prove a general product theorem for affine manifolds, due essentially to Kashiwabara [2]. It is worth noting that this proof remains valid in the appropriate infinite dimensional setting.

The results of § 3 essentially contain the global part of the de Rham theorem; in § 4 we investigate the local question for an arbitrary metric connection.

In [7] H. Wu used the Ambrose-Singer theorem on holonomy and the Ambrose-Hicks theorem to prove the de Rham theorem; the global part of his proof is thus really contained in the Ambrose-Hicks theorem. Since O'Neill's construction gives a simple proof of the Ambrose-Hicks theorem, it is not surprising that it also works for the de Rham theorem.

2. Construction of affine and semi-Riemannian coverings

In this section we follow O'Neill's exposition in [4] as closely as possible, making changes only where necessary to accommodate the more general setting. We omit or merely outline proofs which are essentially the same as in [4].

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The basic difference between the affine and semi-Riemannian cases on the one hand, and the Riemannian case on the other, lies in the unavailability of the Hopf-Rinow theorem; so that geodesics used by O'Neill have to be replaced systematically by finitely broken geodesics in the arguments which follow. (It should be noted that the Hopf-Rinow theorem is actually superfluous even in the Riemannian case, for the applications we have in mind.)

First, we list some well-known facts about affine manifolds and maps needed in the sequel. Let M and N be C^∞ manifolds equipped with affine connections ∇ and ∇' respectively. An *affine* (or *connection-preserving*) map $\phi\colon M \to N$ is a map such that if vector fields X, Y on M are respectively ϕ-related to vector fields X', Y' on N, then $\nabla_X Y$ is ϕ-related to $\nabla'_{X'} Y'$. Alternatively, ϕ_* commutes with parallel translation along curves. It follows that ϕ maps geodesics onto geodesics and is smooth. Each point p in an affine manifold M has a convex normal neighborhood $U(p)$. That is, $U(p)$ is the diffeomorphic image under $\exp_p$ of an open ball in M_p, the tangent space of M at p; and any two points in $U(p)$ can be joined by a unique geodesic segment lying entirely within $U(p)$. If $\phi\colon M \to N$ is an affine local diffeomorphism, it follows that ϕ maps sufficiently small convex normal balls diffeomorphically onto convex normal balls, and therefore that ϕ is uniquely determined by $(\phi_*)_p$ for any $p \in M$ if M is connected.

An affine manifold is said to be *complete* (or *geodesically complete*) if each geodesic γ is defined on the entire real line.

We will need the following lemma, appearing in Hicks [1, Theorem 3], and outline a proof indicating explicitly the role played by broken geodesics.

Lemma 1. *Let M and N be n-dimensional connected C^∞ manifolds each carrying affine connections. Let M be complete, and ϕ be a connection-preserving local diffeomorphism of M into N. Then M is a covering space of N.*

Proof. First, we show that ϕ is onto. Let $p \in M$. Then $\phi(p)$ can be joined to an arbitrary point $q \in N$ by a broken geodesic γ with k breaks, for some k. We show that γ can always be lifted to a broken geodesic $\tilde{\gamma}$ covering it (i.e., $\phi \circ \tilde{\gamma} = \gamma$). Hence if $q = \gamma(t_0)$, then $q = \phi(\tilde{\gamma}(t_0)) \in \phi(M)$. If $k = 1$ then γ is a geodesic. Suppose $\gamma(0) = \phi(p), \gamma'(0) = w$. Then the geodesic $\tilde{\gamma}$ in M satisfying $\tilde{\gamma}(0) = p, \tilde{\gamma}'(0) = v$, where $\phi_* v = w$, covers γ (it is definable for all t, by completeness of M). The proof is completed by successively lifting each segment of γ to the endpoint of the preceding lifted segment. It follows also that N is complete.

Finally, any (convex) normal ball in N is evenly covered by ϕ, as is easily shown.

Now we give the affine version of the construction in [4]. The semi-Riemannian case is practically identical (see § 4). Simply replace "affine diffeomorphism" by "isometry" where appropriate.

Let $U = \{U_\alpha \mid \alpha \in A\}$ be an indexed collection of subsets of some set. By a *semiequivalence relation* on the index set A we mean a reflexive, symmetric relation $\sim$ such that (1) $\alpha \sim \beta$ and $\beta \sim \gamma$ imply $\alpha \sim \gamma$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ and (2) $\alpha \sim \beta$ in A implies $U_\alpha \cap U_\beta \neq \emptyset$.

Remark. If all the above conditions hold with the exception of (2), we can define a new relation by suppressing all relations $\alpha \sim \beta$ if $U_\alpha \cap U_\beta = \emptyset$; the new relation obtained is then a semiequivalence.

Proposition 1. *Let $U = \{U_\alpha \mid \alpha \in A\}$ be an open covering of an affine manifold M, and $\sim$ be a semiequivalence relation on A. Then there exist* (1) *an affine manifold X,* (2) *an affine local diffeomorphism $\psi: X \to M$, and* (3) *for each $\alpha \in A$, a cross-section $\lambda_\alpha: U_\alpha \to X$ of ψ on U_α such that $\lambda_\alpha = \lambda_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$ if and only if $\alpha \sim \beta$ in A.*

Proposition 2. *With hypotheses and notation as in Proposition* 1, *let $\{\phi_\alpha: U_\alpha \to N \mid \alpha \in A\}$ be a collection of affine local diffeomorphisms into an affine manifold N. If $\alpha \sim \beta$ in A implies $\phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$, then there exists a unique affine local diffeomorphism $\phi: X \to N$ such that $\phi \circ \lambda_\alpha = \phi_\alpha$ for all $\alpha \in A$.*

Remarks concerning Proposition 2. (1) If the U_α are assumed to be convex, then $U_\alpha \cap U_\beta$ are connected. It follows that if $\alpha \sim \beta$ implies $(\phi_\alpha)_*(p_\alpha) = (\phi_\beta)_*(p_\alpha)$ for some $p_\alpha \in U_\alpha \cap U_\beta$, then the conclusion of Proposition 2 holds. (2) If the U_α are assumed to be convex, $\phi_\alpha: U_\alpha \to N$ are given, but there is no semiequivalence relation on the index set A, then setting $\alpha \sim \beta$ if $\varphi_\alpha = \varphi_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$ gives a semiequivalence, and the conclusion of Propositions 1 and 2 are valid.

The proofs of Propositions 1 and 2 are essentially identical to those given in [4]. The idea, of course, is to construct X by gluing together disjoint copies of the U_α according to the following rule: U_α and U_β are glued together byidentifying the two distinct copies of $U_\alpha \cap U_\beta$ which they contain, *provided $\alpha \sim \beta$.*

Even if M is connected, X need not be. However, define a *chain* in the index set A of U to be a finite sequence $\alpha_1 \sim \cdots \sim \alpha_n$ of successively related indices. Then both X and M are connected, if the elements of U are connected and any two elements of A are chainable.

We now find a criterion for the completeness of the manifold X. A chain $\alpha_1 \sim \cdots \sim \alpha_n$ in A *covers* a curve segment $\sigma: [0, b] \to M$ provided there exist numbers $0 = t_0 < t_1 < \cdots < t_n = b$ such that $\sigma \mid [t_{i-1}, t_i]$ lies in U_{α_i} for $1 \leq i \leq n$. Then we say that *$(U, \sim)$ is extendable from a point $p \in U_\alpha$ by broken geodesics* provided that any broken geodesic $\gamma: [0, b] \to M$ such that $\gamma(0) = p$ can be covered by a chain $\alpha = \alpha_1 \sim \cdots \sim \alpha_n$ in the index set of U.

Proposition 3. *Let $U = \{U_\alpha \mid \alpha \in A\}$ be an open covering of a connected affine manifold M, with $\sim$ a semiequivalence relation on A. If M is complete and $(U, \sim)$ is extendable from $p \in U_\alpha$ by broken geodesics, then in X the component C containing the point $\lambda_\alpha(p)$ is complete, and hence $\psi \mid C: C \to M$ is an affine covering.*

Proof. Let $\beta: [0, b) \to C$ be a geodesic in C such that $\beta(0) = \lambda_\alpha(q)$. It suffices to show that β has an extension past b. Let γ be a broken geodesic joining p to q (by connectedness of M); now $\psi \circ \beta$ is a geodesic starting at q by affineness of ψ, and by completeness of M it can be extended to a geodesic δ: $[0, b] \to M$. Now by hypothesis the broken geodesic $\gamma * \delta$ (* denotes curve multiplication) can be covered by a chain $\alpha = \alpha_1 \sim \cdots \sim \alpha_n$, and hence in particular δ can be covered by a chain $\alpha_m \sim \cdots \sim \alpha_n$. That is, there are numbers t_i such that $\delta | [t_{i-1}, t_i]$ lies in the domain U_{α_i} of λ_{α_i} $(m \leq i \leq n)$. Thus we have well-defined geodesics $\lambda_{\alpha_i} \circ \delta$: $[t_{i-1}, t_i] \to X$. Since $\alpha_{i-1} \sim \alpha_i$ we have $\lambda_{\alpha_{i-1}} = \lambda_{\alpha_i}$ on $U_{\alpha_{i-1}} \cap U_{\alpha_i}$; hence these segments constitute a single geodesic segment $\tilde{\beta}$: $[0, b] \to X$. By construction, β and $\tilde{\beta}$ are initially the same, hence $\tilde{\beta}$ provides the required geodesic extension of β to (and thus past) b. The final assertion in the proposition follows from Lemma 1.

Now, as a preliminary application of the preceding theory, we give a proof of the Ambrose-Hicks theorem on parallel translation of curvature [1]. Again we follow O'Neill as closely as possible, but our notation is partially inherited from J. A. Wolf [6].

We will need the following notational conventions. If p is a point in a complete affine manifold M, and $v \in M_p$, then let $\gamma_t(v)$: $[0, 1] \to M$ be the uniquely defined geodesic segment satisfying $\gamma_0(v) = p$, $\gamma_0'(v) = v$. Given tangent vectors $v_1, \cdots, v_k$ at p the broken geodesic $\gamma(t) = \gamma_t(v_1, \cdots, v_k)$: $[0, k] \to M$ emanating from p is defined by: $\gamma(t) = \gamma_t(v_1)$ for $t \in [0, 1]$; $\gamma(t) = \gamma_{t-i}[\tau(v_1, \cdots, v_i)v_{i+1}]$ for $t \in [i, i+1], 1 \leq i \leq k$, where $\tau(v_1, \cdots, v_i)$: $M_p \to M_{\gamma(i)}$ denotes parallel translation along $\gamma_t(v_1, \cdots, v_i)$ from the initial point $\gamma(0) = p$ to the final point $\gamma(i)$. Thus the $(i+1)$-st segment of γ is the geodesic δ: $[0, 1] \to M$ with $\delta(0) = \gamma(i)$ and $\delta'(0) = \tau(v_1, \cdots, v_i)v_{i+1}$.

Theorem (*Ambrose-Hicks*). *Let M and N be complete n-dimensional affine manifolds; let $p \in M$ and $q \in N$; let l: $M_p \to N_q$ be a nonsingular linear map. Suppose that for each $v_1, \cdots, v_k \in M_p$ the nonsingular linear map $L = l(v_1, \cdots, v_k) = \tau(lv_1, \cdots, lv_k) \circ l \circ \tau^{-1}(v_1, \cdots, v_k)$ maps the curvature and torsion tensors of M on those of N,* i.e., $R'_{LvLw}Lu = LR_{vw}u$, $T'(Lv, Lw) = LT(v, w)$ *for $u, v, w \in M_{\gamma(k)}$, where $R(R')$, $T(T')$ denote the curvature and torsion tensors of $M(N)$. Then there exist* (1) *a complete affine manifold X,* (2) *affine coverings ψ: $X \to M$ and ϕ: $X \to N$, and* (3) *a point $x \in X$ such that $\psi(x) = p, \phi(x) = q$, and $l \circ \psi_* = \phi_*$ at x.*

Thus if M (also N) is simply connected, then $\phi \circ \psi^{-1}$: $M \to N$ is an affine covering (diffeomorphism) with differential map l at p.

Proof. For each $v_1, \cdots, v_k \in M_p$, let $U(v_1, \cdots, v_k)$ be a convex normal ball at $\gamma_k(v_1, \cdots, v_k)$. Consider the nonsingular linear map $l(v_1, \cdots, v_k) = \tau(lv_1, \cdots, lv_k) \circ l \circ \tau^{-1}(v_1, \cdots, v_k)$ from $M_{\gamma_k(v_1, \cdots, v_k)}$ to $N_{\gamma_k(lv_1, \cdots, lv_k)}$. Since $l(v_1, \cdots, v_k, v_{k+1})$ preserves curvature and torsion for all v_{k+1}, it follows that $l(v_1, \cdots, v_k)$ and $U(v_1, \cdots, v_k)$ satisfy the hypotheses of the Cartan Lemma, stated below. Its conclusion then gives a unique affine local diffeomorphism $\phi(v_1, \cdots, v_k)$: $U(v_1, \cdots, v_k) \to N$ such that $\phi(v_1, \cdots, v_k)_*$ at $\gamma_k(v_1, \cdots, v_k)$ is $l(v_1, \cdots, v_k)$.

Thus $U = \{U(v_1, \cdots, v_k) \mid v_i \in M_p, k = 1, 2, \cdots\}$ is a convex open covering of M, and we define a relation $\sim$ on its index set by: $(v_1, \cdots, v_k) \sim (w_1, \cdots, w_k)$ if $\phi(v_1, \cdots, v_k) = \phi(w_1, \cdots, w_k)$ on $U(v_1, \cdots, v_k) \cap U(w_1, \cdots, w_k) \neq \emptyset$. By the second remark following Proposition 2, $\sim$ is a semiequivalence.

We assert that $(U, \sim)$ is extendable from $p \in U(0)$ by broken geodesics, where $0 \in M_p$. Let γ: $[0, k] \to M$ be a broken geodesic such that $\gamma(0) = p$. Cover γ by a finite number of overlapping neighborhoods from U, which we can assume are of the form $U_0 = U(0)$, $U_1 = U(v_1)$, $U_2 = U(v_1, v_2)$, $\cdots$, $U_l = U(v_1, \cdots, v_l)$ for some $v_i \in M_p$. Then we have affine local diffeomorphisms $\phi_i = \phi(v_1, \cdots, v_i)$: $U_i \to N$, $1 \leq i \leq l$. Now choose numbers $t_i \in [0, 1]$ such that $p_i = \gamma(i - 1 + t_i) \in U_{i-1} \cap U_i$. We assert that $(\phi_{i-1})_{*p_i} = (\phi_i)_{*p_i}$. This is because the differentials of ϕ_{i-1} and ϕ_i commute with parallel translation, so $(\phi_i)_{*p_i} = l(v_1, \cdots, v_{i-1}, t_i v_i)$ by construction of l since $(\phi_i)_{*\gamma(i)} = l(v_1, \cdots, v_i)$. Similarly, $(\phi_{i-1})_* \gamma(i-1) = l(v_1, \cdots, v_{i-1})$, hence again $(\phi_{i-1})_{*p_i} = l(v_1, \cdots, v_{i-1}, t_i v_i)$. It follows by the first remark following Proposition 2 that $\phi_{i-1} = \phi_i$ on $U_{i-1} \cap U_i$, and hence $U_{i-1} \sim U_i$ as required.

This argument also shows that any index $(v_1, \cdots, v_k)$ is chainable to $0 \in M_p$. The theorem follows now by applying Propositions 2 and 3.

Lemma 2 (*Cartan*). *Let U be a normal ball at a point p of an n-dimensional affine manifold M, N be a complete n-dimensional affine manifold, and l: $M_p \to N_q$ be a nonsingular linear map. If $l(w) = \tau(lw) \circ l \circ \tau(w)^{-1}$, considered as a map between the tensor algebras of the tangent spaces concerned, maps the curvature and torsion tensors of M onto those of N for each $w \in M_p$ such that $\exp_p w \in U$, then there exists a unique affine local diffeomorphism ϕ: $U \to N$ such that $(\phi_*)_p = l$.*

For a proof, see e.g. [6, pp. 27–30].

3. A global product theorem

In this section we use the method of § 2 to prove the following result for affine manifolds, obtained by Kashiwabara [2] using homotopy methods. Again the semi-Riemannian version is almost identical and is omitted (see § 4).

Theorem. *Let M be a simply-connected complete affine manifold, and suppose that there exist two globally defined complementary parallel fields of planes T_1 and T_2. Suppose further that the T_i are completely integrable, and that for each $p \in M$ the leaves $M_i(p)$ of T_i through p give a local affine product structure, i.e., there exist neighborhoods U_i of p in $M_i(p)$ and U of p in M such that U is affinely diffeomorphic to $U_1 \times U_2$. Then M is affinely diffeomorphic to the affine product $M_1 \times M_2$.*

Before giving the proof, we recall a few facts about affine product structures which will be needed.

First, a submanifold N of an affine manifold $\bar{N}$ with connection $\bar{\nabla}$ is said to be *autoparallel* if tangency to N is preserved by parallel translation along curves in N. It follows that if X, Y are vector fields on N, and $\bar{X}, \bar{Y}$ are extensions to $\bar{N}$, then there is a naturally induced connection on N defined by $\nabla_Y X = \bar{\nabla}_{\bar{Y}} \bar{X}$. It follows also that N is totally geodesic, i.e., geodesics of N are geodesics of $\bar{N}$.

Now, if N_1 and N_2 are affine manifolds with connections ∇^1 and ∇^2, then there is a connection ∇, the *product* connection, on $N_1 \times N_2 = N$ such that the tangent spaces of the submanifolds of form $N_1 \times q$ (respectively $p \times N_2$) through $(p, q) \in N_1 \times N_2$ are parallel (hence $N_1 \times q$ (respectively $p \times N_2$) are autoparallel for all $(p, q) \in N_1 \times N_2$), and ∇ induces connections $\nabla^1(q)$ and $\nabla^2(p)$ on $N_1 \times q$ and $p \times N_2$ respectively such that the diffeomorphisms $N_1 \to N_1 \times q$ and $N_2 \to p \times N_2$ are affine.

In order to define ∇ we need the following remarks. The tangent space $(N_1 \times N_2)_{(p,q)}$ is the direct sum $(N_1)_p + (N_2)_q$. If $v_1 \in (N_1)_p$ and $v_2 \in (N_2)_q$, we write (v_1, v_2) or $v_1 + v_2$ to denote their sum in $(N_1 \times N_2)_{(p,q)}$. It follows that the module $\mathfrak{X}(N_1 \times N_2)$ of C^∞ vector fields on $N_1 \times N_2$ consists of elements of form (fX_1, gX_2) where $f, g \in \mathfrak{F}(N_1 \times N_2)$, the ring of C^∞ functions on $N_1 \times N_2$, and $X_i \in \mathfrak{X}(N_i)$, the set of C^∞ vector fields on N_i. Note that the mappings $X_1 \to (X_1, 0)$, resp. $X_2 \to (0, X_2)$ imbed $\mathfrak{X}(N_1)$, resp. $\mathfrak{X}(N_2)$, in $\mathfrak{X}(N_1 \times N_2)$. Under this identification $\mathfrak{X}(N_1)$ and $\mathfrak{X}(N_2)$ generate $\mathfrak{X}(N_1 \times N_2)$ over the ring $\mathfrak{F}(N_1 \times N_2)$. By a well-known theorem [3, pp. 25 and 30] it suffices to define $\nabla_{(v_1, v_2)}$ on $\mathfrak{F}(N_1 \times N_2)$ and on $\mathfrak{X}(N_1)$ and $\mathfrak{X}(N_2)$, and then extend $\nabla_{(v_1, v_2)}$ to a derivation on the set of tensor fields of $N_1 \times N_2$. To this end, set $\nabla_{(v_1, v_2)} f = (v_1, v_2) f$ for $f \in \mathfrak{F}(N_1 \times N_2)$, and $\nabla_{(v_1, v_2)} X_1 = \nabla^1_{v_1} X_1, \nabla_{(v_1, v_2)} X_2 = \nabla^2_{v_2} X_2$. It is immediate that ∇ has the properties of the preceding paragraph.

It follows that if α_i are curves in N_i and X_i are parallel vector fields along α_i, then (X_1, X_2) is parallel along (α_1, α_2) and conversely. In particular (α_1, α_2) is a geodesic of $N_1 \times N_2$ if and only if α_i are geodesics in N_i.

If $(x_1, \cdots, x_r)$ and $(x_{r+1}, \cdots, x_{r+s})$, $r + s = n$, are local coordinates in neighborhoods U_1 of $p \in N_1$ and U_2 of $q \in N_2$, then we restrict $1 \leq i, j, k \leq r$ to be the early indices, $r + 1 \leq \alpha, \beta, \gamma \leq n$ to be the late indices, and $1 \leq I, J, K \leq n$ will denote general indices, so that $X_i = \partial/\partial x^i$ and $X_\alpha = \partial/\partial x^\alpha$ span $\mathfrak{X}(U_1 \times U_2)$ and the connection ∇ is determined on $U_1 \times U_2$ by $\nabla_{X_I} X_J = \sum \Gamma_{IJ}^K X_K$. It follows immediately that the Γ_{IJ}^K as well as $(\partial/\partial X_L)\Gamma_{IJ}^K$ vanish if I, J, K, L are not all early or all late, and this condition on the Γ_{IJ}^K characterizes the product connection on a product manifold. The components T_{IJ}^K and R_{IJK}^L of the torsion and curvature tensors satisfy the same condition.

Let $v_i \in (N_1)_p, w_i \in (N_2)_q$ for $i = 1, 2, \cdots, k$. Then the broken geodesics $\gamma_t(v_1, \cdots, v_k, w_1, \cdots, w_k) = \gamma(t), \gamma_t(w_1, \cdots, w_k, v_1, \cdots, v_k) = \bar{\gamma}(t)$ issuing from (p, q) have the same endpoints $\gamma(2k) = \bar{\gamma}(2k)$. This is because $\gamma_k(v_1 + w_1, \cdots, v_k + w_k) = (\gamma_k(v_1, \cdots, v_k), \gamma_k(w_1, \cdots, w_k)) = \gamma_{2k}(v_1, \cdots, v_k, w_1, \cdots, w_k) = \gamma_{2k}(w_1, \cdots, w_k, v_1, \cdots, v_k)$. Also parallel translation along the broken geodesic $\gamma(t)$ agrees with that along $\bar{\gamma}(t)$, since they both agree with that along $\gamma_t(v_1 + w_1, \cdots, v_k + w_k)$.

Proof of theorem. Fix $p \in M$. By definition the leaves $M_i = M_i(p)$ are autoparallel in M, and hence totally geodesic. It follows easily that they are complete. We will set up local affine diffeomorphisms φ_α defined on convex normal balls U_α in $M_1 \times M_2$, define a semiequivalence relation on the index α and apply the results of § 2.

For convenience, we will say that curves, tangent vectors, etc., tangent to $T_1(T_2)$ in M, or parallel to $M_1(M_2)$ in $M_1 \times M_2$, are horizontal (vertical). A vector tangent to M will be denoted by (v, w) or $v + w$ where v is horizontal, and w vertical.

We now prove the following lemmas, generalizing similar statements for $M_1 \times M_2$.

Lemma 1. *Given a horizontal curve $\gamma_t\colon [0, k] \to M$, there exist (1) a vertical CNN (convex normal neighborhood) $U_2(\gamma_0)$ in $M_2(\gamma_0)$ which is affinely diffeomorphic to a vertical* CNN *(convex normal neighborhood) $U_2(\gamma_k)$ of γ_k in $M_2(\gamma_k)$,* (2) *a neighborhood $U_1(\gamma)$ of γ in $M_1(\gamma_0)$ such that $U_1(\gamma) \times U_2(\gamma_0)$ is affine diffeomorphic to a neighborhood of γ. The same statements hold with horizontal, vertical, and* 1, 2 *interchanged.*

Proof. Cover γ with a finite number of CNPN (convex normal product neighborhoods) $U_i = U_1(\gamma_{t_i}) \times U_2(\gamma_{t_i})$, $0 = t_0 < t_1 < \cdots < t_m = k$, using compactness of $\gamma([0, k])$. On the overlap $U_0 \cap U_1$ the vertical factors of U_0 and U_1 must coincide, so we can assume, by reducing the vertical size of U_0 and U_1 if necessary, that $U_2(\gamma_0)$ is affinely diffeomorphic to $U_2(\gamma_{t_1})$.

Now we repeat the argument on U_1 and U_2, again reducing vertical size if necessary. By again reducing the vertical size of U_0 accordingly, we find $U_2(\gamma_0)$ affinely diffeomorphic to $U_2(\gamma_{t_2})$. After k repetitions of this argument, we find a neighborhood $U_2(\gamma_0)$ (having been reduced in size from the original $U_2(\gamma_0)$) affinely diffeomorphic to $U_2(\gamma_k)$, similarly redefined.

To prove (2), set $U_1(\gamma) = \cup\, U_1(\gamma_{t_i})$. Then $U_1(\gamma) \times U_2(\gamma_0)$ is clearly a PN (product neighborhood) of γ.

Lemma 2. *Let $\gamma(t) = \gamma_t(v_1, \cdots, v_k, w_1, \cdots, w_k)$ and $\bar\gamma(t) = \gamma_t(w_1, \cdots, w_k, v_1, \cdots, v_k)$ be two broken geodesics issuing from p, for $v_i, w_i \in M_p, v_i$ horizontal, w_i vertical. Then the end-points $\gamma(2k) = \bar\gamma(2k)$ coincide.*

Proof. First, note that we can assume that the w_i are small enough to carry out the argument to follow, since we can introduce artificial breaks in γ and $\bar\gamma$ if necessary. This process will change the parametrizations of γ and $\bar\gamma$, but will not affect the paths which they determine.

For each $t \in [0, k]$ consider the horizontal broken geodesic $\delta_s(t) = \gamma_{s+t}(w_1, \cdots, (t-l)w_l, v_1, \cdots, v_k), 0 \le s \le k$, where t lies between integers l and $l + 1$. We apply Lemma 1 to $\delta(t)$ to obtain a vertical CNN $U_2(\delta_0(t))$ of $\delta_0(t)$ such that $U_1(\delta(t)) \times U_2(\delta_0(t))$ is a PN of $\delta(t)$ for some $U_1(\delta(t))$. Now we choose a finite number of CNN $U_2(\delta_0(t_i))$ covering $\gamma(w_1, \cdots, w_k)$ for $0 = t_0 < t_1 < \cdots < t_m = k$. We can arrange that each segment of $\gamma(w_1, \cdots, w_k)$ be contained in one and only one $U_2(\delta_0(t_i))$, by choosing w_i and $U_2(\delta(t_i))$ small enough.

Hence $\gamma(w_1, v_1, \cdots, v_k)$ and $\gamma(v_1, \cdots, v_k, w_1)$ both lie in the PN $U_1(\delta(t_0)) \times U_2(\delta_0(t_0))$; hence $\gamma_{k+1}(w_1, v_1, \cdots, v_k) = \gamma_{k+1}(v_1, \cdots, v_k, w_1)$, and since parallel translation along these two broken geodesics agrees, $\gamma_t(w_1, v_1, \cdots, v_k, w_2, \cdots w_k) = \gamma_t(v_1, \cdots, v_k, w_1, \cdots, w_k)$ for $k + 1 \leq t \leq 2k$. Now we repeat the reasoning on the two once-broken geodesics $\gamma_t(w_1, w_2, v_1, \cdots, v_k)$ and $\gamma_t(w_1, v_1, \cdots, v_k, w_2)$, $1 \leq t \leq k + 2$, lying in the product neighborhood $U_1(\delta(t_1)) \times U_2(\delta_0(t_2))$, to find $\gamma_{2k}(w_1, w_2, v_1, \cdots, v_k, w_3, \cdots, w_k) = \gamma_{2k}(w_1, v_1, \cdots, v_k, w_2, \cdots, w_k)$. Repeating this argument proves Lemma 2.

Lemma 3. *Let $\gamma(v_1, \cdots, v_k, w_1, \cdots, w_k)$ be a broken geodesic with $v_i \in M_p$ horizontal and $w_i \in M_p$ vertical. Then there exists a* CNN *$U(v_1, \cdots, v_k, w_1, \cdots, w_k)$ which is affinely diffeomorphic to the affine product $U_1(v_1, \cdots, v_k) \times U_2(w_1, \cdots, w_k)$, where $U_1(v_1, \cdots, v_k)$ is a* CNN *of $\gamma_k(v_1, \cdots, v_k)$ in $M_1 = M_1(p)$ and $U_2(w_1, \cdots, w_k)$ is a* CNN *of $\gamma_k(w_1, \cdots, w_k)$ in $M_2(p) = M_2$.*

Proof. Combine Lemmas 1 and 2.

Now let $L_i: M_i \to M$ be the affine diffeomorphisms identifying M_i with $M_i(p)$ in M, and let $\tilde{p} = (p_1, p_2) \in M_1 \times M_2$ be the point such that $L_i(p_i) = p \in M$. By the local product assumption on M, there exist CNN U_i of p_i in M_i such that $L = L_1 \times L_2: U_1 \times U_2 \to M$ is an affine product diffeomorphism. Set $l = (L_*)_{\tilde{p}}$. Note that $L_1(\gamma(v_1, \cdots, v_k)) = \gamma(lv_1, \cdots, lv_k), L_2(\gamma(w_1, \cdots, w_k)) = \gamma(lw_1, \cdots, lw_k)$ for $v_i \in (M_1)_{p_1}, w_i \in (M_2)_{p_2}$. Now each point (q_1, q_2) of $M_1 \times M_2$ is of form $\gamma_k(v_1 + w_1, \cdots, v_k + w_k) = \gamma_{2k}(v_1, \cdots, v_k, w_1, \cdots, w_k)$ for some set α of k vectors $(v_i, w_i) \in (M_1 \times M_2)_{\tilde{p}}$ for some k. Furthermore, (q_1, q_2) has a CNN $U_\alpha = U(v_1, \cdots, v_k, w_1, \cdots, w_k)$ of form $U_1(v_1, \cdots, v_k) \times U_2(w_1, \cdots, w_k)$ where $U_1(v_1, \cdots, v_k)$ and $U_2(w_1, \cdots, w_k)$ are respectively CNN of $\gamma_k(v_1, \cdots, v_k)$ in M_1 and $\gamma_k(w_1, \cdots, w_k)$ in M_2; we can take U_α small enough so that $L_1(U_1(v_1, \cdots, v_k)) = U_1(lv_1, \cdots, lv_k)$ and $L_2(U_2(w_1, \cdots, w_k)) = U_2(lw_1, \cdots, lw_k)$ are the CNN guaranteed by Lemma 3 for the broken geodesic $\gamma(lv_1, \cdots, lv_k, lw_1, \cdots, lw_k)$. Now let φ_α map $U_\alpha = U_1(v_1, \cdots, v_k) \times U_2(w_1, \cdots, w_k)$ as the product map $L_1 \times L_2$ onto $U_1(lv_1, \cdots, lv_k) \times U_2(lw_1, \cdots, lw_k)$. φ_α is an affine diffeomorphism since the L_i are, and φ_α maps the CNPN U_α onto a local CNPN in M. It follows that if $\varphi_\alpha(q_1, q_2) = \varphi_\beta(q_1, q_2)$ for some $(q_1, q_2) \in M_1 \times M_2$, then the φ_α agrees with φ_β on some CNPN $U_\alpha \cap U_\beta$ of (q_1, q_2), and hence the φ_α induce a semiequivalence relation on the U_α by $\alpha \sim \beta$ if $\varphi_\alpha = \varphi_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$. Note that the U_α cover $M_1 \times M_2$. Furthermore $\gamma_t(v_1 + w_1, \cdots, v_k + w_k)$ is extendable from (p_1, p_2) by covering it with a finite number of sufficiently small CNPN U_i. So the affine manifold X with affine covering maps $\psi: X \to M_1 \times M_2$ and $\varphi: X \to M$ is defined; furthermore M is simply connected so X is affinely diffeomorphic to M. It follows that $\psi: M \to M_1 \times M_2$ is the affine universal covering of $M_1 \times M_2$. But then M is affinely diffeomorphic to $\hat{M}_1 \times \hat{M}_2$, $\hat{M}_i$ is the affine universal covering of M_i, and $\psi = \psi_1 \times \psi_2$ where $\psi_i: \hat{M}_i \to M_i$ are the universal covering maps.

Now $M = \hat{M}_1 \times \hat{M}_2$ has (possibly) another set of complementary autoparallel integrable distributions T_1 and T_2 by assumption. We must show that these two sets of local product structures coincide so that $M_i = M_i(p) = \hat{M}_i$. By autoparallelism it suffices to show they agree at one point. We use the relation $\varphi \circ \lambda_\alpha = \varphi_\alpha$ of Proposition 1, § 1, where λ_α is a local cross-section on U_α of $\psi_1 \times \psi_2$. Take $\alpha = \{0\}$, $0 \in (M_1 \times M_2)_{\tilde{p}}$. Then $\varphi_\alpha = L = L_1 \times L_2$ on a CNN $U_1 \times U_2$ of $\tilde{p}$. Since λ_α is a cross-section for $\psi_1 \times \psi_2$, λ_α maps $U_1 \times U_2$ affinely diffeomorphically onto a CNN $\hat{U}_1 \times \hat{U}_2$ in $\hat{M}_1 \times \hat{M}_2$. Now $\varphi \circ \lambda_\alpha = \varphi_\alpha$ implies $\varphi(\hat{U}_1 \times \hat{U}_2) = L(U_1 \times U_2)$, showing that the two sets of foliations agree at $p \in M$. Hence $M = M_1 \times M_2$.

4. The local product theory and the de Rham theorem

In this section we prove the following generalization of the de Rham product decomposition theorem for Riemannian manifolds, which also includes the semi-Riemannian version by H. Wu.

Theorem. *Suppose that M is a simply-connected semi-Riemannian manifold with a complete metric connection ∇, and that for $p \in M$, there exists a nondegenerate proper subspace $T_1(p)$ of M_p with the following property. For every curve τ: $[a, b] \to M$ with $\tau(a) = p$, $\tau^{-1}R_{\tau v, \tau w} \circ \tau$ and $\langle \tau^{-1}T(\tau v, \tau w), u\rangle$ vanish unless $u, v, w \in T_1(p)$ or $u, v, w \in T_1{}^{\perp}(p) = T_2(p)$, where τ also denotes parallel translation along τ from $\tau(a)$ to $\tau(b)$, R and T denote the curvature and torsion tensor of M respectively.*

Then M is isometric and affinely diffeomorphic to the semi-Riemannian product manifold $M_1 \times M_2$ with the product affine connection, where $M_i = M_i(p)$ are the maximal integral manifolds through p of the mutually orthogonal autoparallel fields of planes T_1 and T_2 generated by $T_1(p)$ and $T_2(p)$, in the induced metrics and connections.

Remark. The condition on the curvature is equivalent to: the linear holonomy group $\Phi(p)$ at p leaves invariant the subspace $T_1(p)$ (hence also $T_2(p)$).

Before giving the proof, we will need to extend the results of §§ 2 and 3 to the case of semi-Riemannian manifolds and metric connections. A C^∞ manifold M equipped with a nondegenerate metric tensor (inner product) $\langle\ ,\ \rangle$ defined on each tangent space, and such that $\langle X, Y\rangle \in C^\infty$ if X, Y are C^∞ vector fields on M, is called a C^∞ semi-Riemannian manifold. An affine connection ∇ on M satisfying $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, X, Y, Z \in \mathfrak{X}(M)$, is called a *metric connection* for $\langle\ ,\ \rangle$. Geometrically, this condition means that parallel translations along curves in M are isometries with respect to the metric tensor. If in addition, the torsion tensor $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ vanishes, then ∇ is called the Levi-Civita connection associated with ∇. The Levi-Civita connection exists, is unique for any semi-Riemannian manifold, and is given by the formula

$$2\langle \nabla_x Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle .$$

Note that nondegeneracy of the inner product is needed in order that $\nabla_X Y$ be well-defined by this formula.

A semi-Riemannian manifold, unless further qualified, will be assumed from now on to carry its Levi-Civita connection. A semi-Riemannian manifold with a metric connection will be called a metrically connected manifold. The appropriate structure-preserving map for semi-Riemannian manifolds is the isometry. An isometry is a mapping $\varphi\colon M \to N$ satisfying $\langle \varphi_* v, \varphi_* w\rangle' = \langle v, w\rangle$, where $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ denote the metric tensors of M and N respectively, and $v, w \in M_p$ for any $p \in M$. If M and N are both n-dimensional semi-Riemannian manifolds, then every isometry $\varphi\colon M \to N$ is also affine, as is shown using the formula for the Levi-Civita connection. φ is also a local diffeomorphism, and if φ is bijective as well, then φ is said to be *isometric*, and M and N are *isometric* (equivalent). Since an isometry φ is affine, it is also *geodesic*: if γ is a geodesic of M, then $\varphi\circ\gamma$ is a geodesic of N. The converse can also be shown: a geodesic map φ with φ_* injective is affine. On the other hand, an affine map $\psi\colon M \to N$ for semi-Riemannian M and N is not necessarily isometric; for example, a homothety, a map ψ satisfying $\langle \psi_* v, \psi_* w\rangle = K\langle v, w\rangle$ where K is a constant, is affine, as is seen again by using the formula for the Levi-Cività connection. However, if an affine map is an isometry at one point, it is an isometry (see Lemma 1).

If M and N have arbitrary metric connections, then the situation is more complicated. An isometry $\varphi\colon M \to N$ is not necessarily affine; and as before, an affine map is not necessarily isometric. The appropriate structure-preserving map in this case must therefore be a map φ which is at once an isometry and affine with respect to the metric connections. We will say that φ is *metric-affine*. Affine maps are still geodesic, but the converse no longer holds. Isometries are also not necessarily geodesic, and geodesity is essential to the techniques of § 2. The following lemma will be sufficient for our purpose:

Lemma 1. *If $\varphi\colon M \to N$ is an affine map, M and N being metrically connected manifolds, then the metric character of φ is completely determined by that of $(\varphi_*)_p$ for any $p \in M$. E.g., if $(\varphi_*)_p$ is an isometry, then φ is metric-affine; if $(\varphi_*)_p$ is a homothety, then φ is a homothety also.*

Proof. If $q \in M$ is arbitrary, let $\alpha\colon [0, 1] \to M$ be a curve with $\alpha(0) = p$, $\alpha(1) = q$. Since M and N are metrically connected, the parallel translations τ and τ' along α and $\phi\circ\alpha$ respectively are isometries. The result follows from the relation $(\varphi_*)_q = \tau'\circ(\varphi_*)_p\circ\tau^{-1}$ expressing the affineness of φ.

Lemma 1 provides an easy way to establish metric analogues of the results of § 2 and § 3. The semi-Riemannian versions of these results are then obtained as special cases. In § 2, the construction is modified as follows: M and N are assumed metrically connected, and the manifold X of Proposition 2.1 is endowed locally with the metric and connection defined on the U_α. X is thus a metrically connected manifold, and ψ and λ_α are metric-affine. If the φ_α in Proposition 2.2 are metric-affine, then $\varphi\colon X \to N$ is also metric-affine. Lemma

2.1 holds in the metric version, since metric-affine maps map sufficiently small convex normal balls geodesically onto convex normal balls when M and N are n-dimensional, as before. Consequently Proposition 2.3 generalizes. We now use Lemma 4.1 to get the metric analogue of the Cartan Lemma. If $l(w)$ satisfies the conditions of Lemma 2.2 for M and N metrically connected, and if $(l_*)_p$ is an isometry, then ϕ is metric-affine. The metric version of the Ambrose-Hicks theorem is now immediate: if L satisfies the conditions of that theorem, and M and N are metrically connected, and in addition if $(l_*)_p$ is an isometry, then φ and ψ are metric-affine. If the connections on M and N are Levi-Cività, then φ and ψ are isometries.

In order to extend the theorem of § 3 and to obtain a de Rham theorem, we now briefly discuss product metrics. If M_1 and M_2 are semi-Riemannian manifolds with inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ respectively, then the *product metric* $\langle\ ,\ \rangle$ is defined by: $\langle (X_1, X_2), (Y_1, Y_2)\rangle = \langle (X_1, Y_1)\rangle_1 + \langle (X_2, Y_2)\rangle_2$. If M_1 and M_2 also carry metric connections ∇^1 and ∇^2, then the product connection ∇ (see § 3) of ∇^1 and ∇^2 is metric for $\langle\ ,\ \rangle$. In particular, the product of Levi-Civita connections is a Levi-Civita connection, as can be verified from the Levi-Civita formula. Using the index conventions for a product chart $(x^1, \cdots, x^n)$ defined on a product neighborhood $U_1 \times U_2$ of $M_1 \times M_2$, as established in § 3, we see that the product metric on $M_1 \times M_2$ is characterized by: $g_{IJ} = 0$ and $(\partial/\partial x^K) g_{IJ} = 0$ if I, J, K are not all early or all late; where $g_{IJ} = \langle \partial/\partial x^I, \partial/\partial x^J\rangle$; and $g_{ij} = \langle \partial/\partial x^i, \partial/\partial x^j\rangle_1$, $g_{\alpha\beta} = \langle \partial/\partial x^\alpha, \partial/\partial x^\beta\rangle_2$ on $U_1 \times q$, $p \times U_2$ respectively for some $(p, q) \in U_1 \times U_2$.

If M is a submanifold of a semi-Riemannian manifold N with inner product $\overline{\langle\ ,\ \rangle}$ and metric connection $\bar{\nabla}$, and if M_p is a nondegenerate subspace of N_p for every $p \in M$ (i.e., $\overline{\langle\ ,\ \rangle}$ restricted to $M_p \times M_p$ is nondegenerate), then $\overline{\langle\ ,\ \rangle}$ induces a metric tensor $\langle\ ,\ \rangle$ on M. If P denotes the tensor field defined on M inducing the orthogonal projection of $N_p \to M_p$ for each $p \in M$, then $(\nabla_X Y)_p = (P \cdot \bar{\nabla}_{\bar{X}} \bar{Y})_p$ defines a metric connection on M for $X, Y \in \mathfrak{X}(M)$ and local extensions $\bar{X}, \bar{Y}$ of X and Y to a neighborhood of p in N. If $\bar{\nabla}$ is a Levi-Civita connection so is ∇. Note that nondegeneracy is essential in order for P and $\langle\ ,\ \rangle$ to be defined.

The theorem of § 3 carries over immediately to the metric case. We must assume that M is metrically connected, the $T_i(p)$ are *nondegenerate* and the neighborhoods U are metric-affinely diffeomorphic to the metric-affine product $U_1 \times U_2$, the U_i equipped with the induced metrics and connections from M. Then M is metric-affinely diffeomorphic to the metric-affine product $M_1 \times M_2$, the M_i with their metrics and connections being induced from M.

The theorem of § 3 is essentially the best possible; an example by Ozeki [3, vol. I, p. 290] shows that even for a torsion-free connection, an affinely connected manifold satisfying all the conditions of that theorem except for the local product condition need not be an affine product manifold. The theorem to be proved shows that under a mild restriction on the torsion, the local product condition can be dispensed with, provided the connection is metric. See [7] for a torsion-free example showing that nondegeneracy of $T_1(p)$ is essential even to get an affine product decomposition; a metric decomposition is strictly speaking out of the question in any case, since M_1 and M_2 inherit degenerate metrics.

Proof of theorem. We show that the conditions of the metric version of the theorem of § 3 are satisfied; the conclusion of that result is the metric-affine product decomposition which we seek.

We first show the equivalence of the condition given in the remark with the condition on the curvature tensor. First assume that the condition on the curvature holds. Let $\{e_I(p)\}$ denote an (orthonormal) base for M_p such that $e_i \in T_1(p)$, $e_\alpha \in T_2(p)$, where, as in § 3, $1 \le I, J, K \le n$, $1 \le i, j, k \le r$, and $r + 1 \le \alpha, \beta, \gamma \le r + s = n$, r and s being the dimensions of $T_1(p)$ and $T_2(p)$ respectively. With respect to this basis, we find (using the curvature symmetries $\langle R_{XY}(Z), W\rangle = -\langle R_{XY}(W), Z\rangle$ and $\langle R_{XY}(Z), W\rangle = \langle R_{ZW}(X), Y\rangle$ valid for metric connections) that the matrix for $\tau^{-1} \circ R_{\tau e_I \tau e_J} \circ \tau$ must be of the form $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ according as both I and J are early or late; otherwise the matrix vanishes. Here A and B are respectively $r \times r$ and $s \times s$ matrices. It follows by the Ambrose-Singer Theorem on holonomy [3, Theorem 9.1, p. 151] that the holonomy Lie algebra, generated by the above matrices, consists of matrices of form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Hence the same is true for the linear holonomy group $\Phi(p)$ (identified with its representation as a group of matrices, induced by $\{e_I(p)\}$). It follows that $\Phi(p)$ leaves $T_1(p)$ and $T_2(p)$ invariant. To prove the converse, note that if $\Phi(p)$ leaves $T_i(p)$ invariant, then it consists of matrices of form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and the holonomy algebra is also of that form, hence $\langle \tau^{-1} R_{\tau e_I, \tau e_J}(\tau e_i), e_\alpha\rangle = 0$. Since the connection is metric, $0 = \langle \tau^{-1} R_{\tau e_I, \tau e_J}(\tau e_i), e_\alpha\rangle = \langle R_{\tau e_I, \tau e_J}(\tau e_i), \tau e_\alpha\rangle = \langle R_{\tau e_i, \tau e_\alpha}(\tau e_I), \tau e_J\rangle$ for all I, J, so $\tau^{-1} R_{\tau e_i, \tau e_\alpha} \circ \tau = 0$.

For the remainder of the proof we follow [3, pp. 180–183], taking the torsion into account. First, we remark that the distributions T_1 and T_2 on M are well-defined at each point $q \in M$ by parallelly translating $T_i(p)$ to the subspace $T_i(q)$ at q using any piecewise C^∞ curve τ joining p and q, by the condition on the holonomy. Note that $T_1(q)$ is orthogonal to $T_2(q)$ since the connection is metric. Differentiability of the T_i follows [3, p. 180], and integrability of T_1 follows from the identity $[X, Y] = \nabla_X Y - \nabla_Y X - T(X, Y)$, where X and Y are taken to be vector fields both lying in T_1. The condition on the torsion guarantees that $T(X, Y) \in T_1$ also, while $\nabla_X Y$ and $\nabla_Y X$ lie in T_1 since if $\alpha(t)$ is the integral curve of X starting from a point q, then $\nabla_X Y = \lim_{t \to 0} (1/t)(\tau_0^t Y(\alpha(t)) - Y_q)$, where τ_0^t denotes parallel translation along α from $\alpha(t)$ to $\alpha(0) = q$. Since $Y(\alpha(t))$ and Y_q belong to T_1, $\nabla_X Y$ does also; similarly for $\nabla_Y X$. T_2 is integrable also by the same reasoning.

Now let $q \in M$ be arbitrary. On a neighborhood V of q there exists a Frobenius coordinate system $(x^1, \cdots, x^n)$ for T_1, i.e., such that $\partial/\partial x^i \in T_1$; and on a neighborhood W of q a Frobenius coordinate system $(y^1, \cdots, y^n)$ for T_2 such that $\partial/\partial y^\alpha \in T_2$. Then on a neighborhood U of q, $(x^1, \cdots, x^r, y^{r+1}, \cdots, y^n)$ form a coordinate system such that $\partial/\partial x^i = X_i$ form a local base for T_1 and $\partial/\partial y^\alpha = X_\alpha$ form a local base for T_2.

Now we have $T(X_i, X_\alpha) = \nabla_{X_i} X_\alpha - \nabla_{X_\alpha} X_i - [X_i, X_\alpha] = \nabla_{X_i} X_\alpha - \nabla_{X_\alpha} X_i$, since $[X_i, X_\alpha] = 0$. But $T(X_i, X_\alpha) = 0$ by the assumption on T, so $\nabla_{X_i} X_\alpha = \nabla_{X_\alpha} X_i$. But $\nabla_{X_i} X_\alpha \in T_2$ while $\nabla_{X_\alpha} X_i \in T_1$ by autoparallelism of the T_i (as in the integrability proof above); and the T_i are mutually orthogonal (or $T_1(q) \cap T_2(q) = \{0\} \subset M_q$), so $\nabla_{X_i} X_\alpha = \nabla_{X_\alpha} X_i = 0$.

(The reasoning used so far holds for non-metric connections as well, provided the hypotheses on the curvature and torsion are changed appropriately, and we assume explicitly that M_p is the direct sum of subspaces $T_1(p)$ and $T_2(p)$ each invariant under $\Phi(p)$. It remains only to establish the existence of local product neighborhoods, and for this the metric is essential.)

We first show that the $g_{IJ} = \langle X_I, X_J \rangle$ have the characteristic properties of a product metric. But $g_{i\alpha} = \langle X_i, X_\alpha \rangle = 0$ since the T_i are mutually orthogonal; and $X_\alpha(g_{ij}) = X_\alpha \langle X_i, X_j \rangle = \langle \nabla_{X_\alpha} X_i, X_j \rangle + \langle X_i, \nabla_{X_\alpha} X_j \rangle = 0$, and similarly for $X_i(g_{\alpha\beta})$. Hence U is isometric with a product neighborhood.

Now we show that $\Gamma^I_{JK} = 0$ and $X_I \Gamma^L_{JK} = 0$ unless I, J, K, L are all early or all late. The first condition on Γ^I_{JK} is immediate, since $\nabla_{X_J} X_K = \Gamma^I_{JK} X_I$ (we will use the Einstein summation convention from now on), $\nabla_{X_\alpha} X_\beta \in T_2$, $\nabla_{X_i} X_j \in T_1$, and $\nabla_{X_i} X_\alpha = \nabla_{X_\alpha} X_i = 0$.

Now let (g^{IJ}) denote the inverse matrix of (g_{IJ}), so that $g_{IK} g^{KJ} = \delta^J_I$. Since (g_{ij}) and $(g_{\alpha\beta})$ are invertible by nondegeneracy of the $T_i(p)$, and $(g_{i\alpha})$ and $(g_{\alpha i})$ vanish, we find that (g^{ij}) and $(g^{\alpha\beta})$ are the inverses of (g_{ij}) and $(g_{\alpha\beta})$, i.e., $g_{ik} g^{kj} = \delta^j_i, g_{\alpha\gamma} g^{\gamma\beta} = \delta^\beta_\alpha$.

With this notation, we can prove the second condition on the Γ^I_{JK}. First, $\nabla_{X_\alpha} \nabla_{X_i} X_j = \nabla_{X_i} \nabla_{X_\alpha} X_j + R_{X_\alpha X_i} X_j$; the curvature term vanishes by assumption, and $\nabla_{X_\alpha} X_j$ vanishes identically on U, so $\nabla_{X_i} \nabla_{X_\alpha} X_j = 0$. Hence $\nabla_{X_\alpha} \nabla_{X_i} X_j = 0$. Now $X_\alpha \langle \nabla_{X_i} X_j, X_k \rangle = \langle \nabla_{X_\alpha} \nabla_{X_i} X_j, X_k \rangle + \langle \nabla_{X_i} X_j, \nabla_{X_\alpha} X_k \rangle = 0$. On the other hand, $X_\alpha \langle \nabla_{X_i} X_j, X_k \rangle = X_\alpha \langle \Gamma^l_{ij} X_l, X_k \rangle = (X_\alpha \Gamma^l_{ij}) g_{lk} + \Gamma^l_{ij} X_\alpha g_{lk}$. The term $\Gamma^l_{ij} X_\alpha g_{lk}$ vanishes, leaving $(X_\alpha \Gamma^l_{ij}) g_{lk} = 0$. Hence $(X_\alpha \Gamma^l_{ij}) g_{lk} g^{km} = X_\alpha \Gamma^l_{ij} \delta^m_l = X_\alpha \Gamma^m_{ij} = 0$. Similarly $X_i \Gamma^\gamma_{\alpha\beta} = 0$. Since Γ^K_{IJ} vanish identically on U for mixed I, J, K, the $X_L \Gamma^K_{IJ}$ have the property required.

We have therefore verified all the hypotheses of the metric-affine version of Kashiwabara's theorem, obtaining a metric-affine generalization of the de Rham theorem.

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